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# Asymptotic behaviour of the Mayer cluster sums for the Ising model in the Bethe approximation

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## Contents



The properties of the high-field polynomials $L_n(u)$, where $u = \exp[-4J/(k_B T)]$, are investigated for the Bethe approximation of the spin $\frac{1}{2}$ Ising model on a lattice which has a coordination number $q$. (The polynomials $L_n(u)$ are essentially lattice gas analogues of the Mayer cluster integrals $b_n(T)$ for a continuum gas.) In particular, a contour integral representation for $L_n(u)$ is established by applying the Lagrange reversion theorem to the implicit equation of state for the Bethe approximation. Various saddle-point methods are then used to analyse the behaviour of the integral representation as $n \to \infty$. In this manner, asymptotic expansions for $L_n(u)$ are obtained which are *uniformly* valid in the intervals $0 < u \leqslant u_c$ and $u_c \leqslant u < 1$, where $u_c = [(\sigma-1)/(\sigma+1)]^2$ is the critical value of the variable $u$, $\sigma \equiv (q-1)$ and $\sigma > 1$. These expansions involve the Airy function $\mathrm{Ai}(z)$ and its first derivative. The high-field polynomial $L_n(u)$ is found to have a trivial zero at $u = 0$, and $n-1$ simple non-trivial zeros $\{u_\nu(\sigma, n); \nu = 1, 2, \ldots, n-1\}$ which are *all* located in the real interval $u_c < u < 1$. An asymptotic expansion for $u_\nu(\sigma, n)$ in powers of $n^{-\frac{2}{3}}$ is derived from the uniform asymptotic representation for $L_n(u)$ which is valid in the interval $u_c \leqslant u < 1$. It is also shown that the *limiting* density of the zeros $\{u_\nu(\sigma, n); \nu = 1, 2, \ldots, n-1\}$ as $n \to \infty$ is given by the simple formula

$$\rho(\sigma, u) = n(2\pi)^{-1}(\sigma+1)\, u^{-1}(u-u_c)^{\frac{1}{2}}(1-u)^{-\frac{1}{2}},$$

where $u_c < u < 1$. Finally, the asymptotic properties of the Bethe polynomial $L_n(u)$ are determined in the mean-field limit $q \to \infty$ and $J \to 0$ with $qJ \equiv J_0$ held constant.

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## 1. Introduction

Consider the spin $\frac{1}{2}$ Ising model of a ferromagnet on a $d$-dimensional lattice $\Omega_d$ with $N$ sites. (For reviews of the Ising model see Domb (1960, 1974).) The hamiltonian for this system is defined to be

$$\mathscr{H} = -J \sum_{(ij)} \sigma_i \sigma_j - m_0 B \sum_{i=1}^{N} \sigma_i, \tag{1.1}$$

where the first summation is over all nearest-neighbour pairs $(ij)$ in the lattice $\Omega_d$, $B$ is the magnetic field, $\sigma_i = \pm 1$ and $J$, $m_0$ are positive constants. In the thermodynamic limit $N \to \infty$ we can write the free energy per spin $g(T, B)$ of the Ising model as

$$-(k_B T)^{-1} g(T,B) = \lim_{N \to \infty} \frac{1}{N} \ln Z_N(T,B), \tag{1.2}$$

where

$$Z_N(T,B) = \sum_{\sigma_1 = \pm 1} \dots \sum_{\sigma_N = \pm 1} \exp[-\mathscr{H}/(k_B T)] \tag{1.3}$$

is the partition function.

It is well known (Domb 1960; Sykes *et al.* 1965) that the free energy $g(T, B)$ can be expanded as a high-field series in the form

$$-(k_B T)^{-1} g(T,B) = -\tfrac{1}{8} q \ln u - \tfrac{1}{2} \ln \mu + \sum_{n=1}^{\infty} L_n(u) \mu^n, \tag{1.4}$$

where

$$u = \exp[-4J/(k_B T)], \tag{1.5}$$

$$\mu = \exp[-2m_0 B/(k_B T)], \tag{1.6}$$

and $q$ is the coordination number of the lattice $\Omega_d$. The coefficient $L_n(u)$ is a polynomial of degree $\frac{1}{2}nq$ in the variable $u$ except when $n$ and $q$ are both odd. (For this special case $L_n(u)$ is a polynomial of degree $nq$ in the variable $u^{\frac{1}{2}}$.) We readily see from (1.4) that the magnetization per spin of the Ising model

$$m = -(\partial g/\partial B)_T, \tag{1.7}$$

also has a high-field series representation

$$m/m_0 = 1 - 2 \sum_{n=1}^{\infty} n L_n(u) \mu^n. \tag{1.8}$$

The high-field polynomials $L_n(u)$ have been determined for reasonably large values of $n$ on various two- and three-dimensional lattices by using graph-theoretic methods (Sykes *et al.* 1965, 1973*a–d*). If the Ising model is interpreted as a model of a lattice gas (Lee & Yang 1952) it is found that, apart from a factor of $u^{nq/2}$, the polynomial $L_n(u)$ is a lattice analogue of the Mayer cluster integral $b_n(T)$ which occurs in the activity expansions for the pressure and density of an imperfect gas.

The asymptotic properties of $L_n(u)$ as $n \to \infty$ are of particular importance in the theories of condensation and critical point phenomena. For example, the behaviour of $L_n(u_c)$ as $n \to \infty$, where $u_c$ is the critical value of the variable $u$, essentially determines the shape of the critical isotherm for the Ising model (Gaunt *et al.* 1964; Gaunt 1967; Gaunt & Sykes 1972). When $u < u_c$ the asymptotic properties of $L_n(u)$ can be used, at least in principle, to investigate the behaviour of the free energy in the neighbourhood of the phase boundary $\mu = 1$ (Fisher 1967; Domb & Guttmann 1970; Domb 1976). Unfortunately, no exact asymptotic analysis of $L_n(u)$ has yet been carried out for $d > 1$.

For the one-dimensional spin $\frac{1}{2}$ Ising model it has been shown (Bessis *et al.* 1976; Joyce 1990) that

$$nL_n(u) = uP_{n-1}^{(1,0)}(1-2u), \tag{1.9}$$

where $P_n^{(\alpha,\beta)}(x)$ denotes a Jacobi polynomial of degree $n$. It follows from this result and the theory of orthogonal polynomials (Szegö 1939) that $u^{-1}L_n(u)$ has $n-1$ simple zeros $\{u_\nu(n); \nu = 1, 2, \ldots, n-1\}$ which all lie in the interval $0 < u < 1$. (For convenience, the zeros $u_\nu(n)$ are arranged in ascending order with $u_\nu(n) < u_{\nu+1}(n)$ for $\nu = 1, 2, \ldots, n-2$.) Joyce (1990) also established uniform asymptotic expansions for $L_n(u)$, which are valid as $n \to \infty$ with $0 < u < 1$, by applying standard techniques to the formula (1.9). These results were then used to derive asymptotic representations for the zeros $u_\nu(n)$. In particular, it was found that

$$u_\nu(n) \sim \left(\frac{j_{1,\nu}}{2n}\right)^2 \left[1 - \frac{j_{1,\nu}^2}{12n^2} + \frac{j_{1,\nu}^2}{720n^4}(-3 + 2j_{1,\nu}^2) + \frac{j_{1,\nu}^2}{20\,160n^6}(40 + 4j_{1,\nu}^2 - j_{1,\nu}^4) + \ldots\right], \tag{1.10}$$

$$u_\nu(n-\nu) \sim 1 - \frac{j_{0,\nu}^2}{4n^2} + \frac{j_{0,\nu}^2}{48n^4}(-2 + j_{0,\nu}^2) + \frac{j_{0,\nu}^2}{2880n^6}(2 + 9j_{0,\nu}^2 - 2j_{0,\nu}^4) + \ldots, \tag{1.11}$$

as $n \to \infty$, with $\nu$ fixed, where $j_{k,\nu}$ denotes the $\nu$th zero of the Bessel function $J_k(z)$.

Our main aim in the present paper is to investigate the exact asymptotic properties of the high-field polynomial $L_n(u)$ for the Bethe approximation (Bethe 1935) of the spin $\frac{1}{2}$ Ising model on a lattice $\Omega_d$ with a coordination number $q$. In the first stage of the analysis a contour integral representation for $L_n(u)$ is derived by applying the Lagrange reversion theorem (Copson 1935) to the implicit equation of state for the Bethe approximation. Next the contour integral for $L_n(u)$ is evaluated asymptotically for large $n$ by using the standard saddle-point method (see Dingle 1973). Hence we obtain a basic asymptotic expansion for $L_n(u)$ which is valid as $n \to \infty$, provided that $u$ is fixed in the interval $0 < u < u_c$, and $q > 2$.

The asymptotic expansion for $L_n(u)$ breaks down completely as $u \to u_c-$ because there are two saddle points which become coincident when $u = u_c$. In §4 we overcome this problem by following the powerful modification of the saddle-point method developed by Chester *et al.* (1957). This procedure yields an asymptotic expansion for $L_n(u)$, in terms of the Airy function $\mathrm{Ai}(z)$ and its first derivative, which is uniformly valid in the interval $0 < u \leqslant u_c$. In §5 we obtain a similar uniform asymptotic expansion for $L_n(u)$ which is valid in the interval $u_c \leqslant u < 1$. It is also shown that the dominant asymptotic behaviour of $L_n(u)$ as $n \to \infty$ and $u \to 1\pm$ is described by the formula

$$L_n(u) \sim (-1)^{n-1} n^{-1} J(\sigma, \eta), \tag{1.12}$$

where

$$\sigma = q - 1, \tag{1.13}$$

$$\eta = (1-u)^{\frac{1}{2}} n, \tag{1.14}$$

and $J(\sigma, \eta)$ is a new transcendental scaling function which reduces to the Bessel function $J_0(2\eta)$ when $\sigma = 1$.

For the Bethe approximation of the spin $\frac{1}{2}$ Ising model it can be shown that the function $u^{-\frac{1}{2}n(\sigma-1)-1} L_n(u)$ is always a polynomial of degree $n-1$ in the variable $u$. The numerical investigations of Bessis *et al.* (1976) and Gaunt (1978) indicate that, for

$\sigma > 1$, this polynomial function has $n-1$ simple zeros $\{u_\nu(\sigma, n); \nu = 1, 2, \ldots, n-1\}$ which are all located in the real interval $u_c < u < 1$. (For convenience, it is assumed that the zeros $u_\nu(\sigma, n)$ are arranged in ascending order with $u_\nu(\sigma, n) < u_{\nu+1}(\sigma, n)$ for $\nu = 1, 2, \ldots, n-2$.) In §6 we use the Airy function representation for $L_n(u)$ to derive an asymptotic expansion for $[u_\nu(\sigma, n) - u_c]/u_c$ in powers of $n^{-\frac{2}{3}}$ which is valid as $n \rightarrow \infty$, with $\nu$ fixed and $\sigma > 1$. The leading-order contribution to this expansion is found to be

$$[u_\nu(\sigma, n) - u_c]/u_c \sim 2^{\frac{4}{3}} \sigma^{\frac{1}{3}} (\sigma^2 - 1)^{-\frac{2}{3}} |a_\nu| \, n^{-\frac{2}{3}}, \tag{1.15}$$

where $a_\nu$ denotes the $\nu$th negative zero of the Airy function Ai$(z)$. We also show that the limiting density of the zeros $\{u_\nu(\sigma, n); \nu = 1, 2, \ldots, n-1\}$ as $n \rightarrow \infty$ is given by the simple formula

$$\rho(\sigma, u) = n(2\pi)^{-1}(\sigma + 1)\, u^{-1}(u - u_c)^{\frac{1}{2}}(1-u)^{-\frac{1}{2}}, \tag{1.16}$$

where $u_c < u < 1$. Finally, in §7 we discuss briefly the asymptotic properties of the high-field polynomials $L_n(u)$ for the Bethe approximation in the mean-field limit $q \rightarrow \infty$, $J \rightarrow 0$ with $qJ \equiv J_0$ held constant.

## 2. Evaluation of $L_n(u)$ for the Bethe approximation

In the Bethe approximation (Bethe 1935; Peierls 1936; Domb 1960) attention is focused on a cluster of spins $C$ which consists of a typical spin $\sigma_0$ and all its nearest-neighbour spins $\sigma_1, \ldots, \sigma_q$. The interaction of the central spin $\sigma_0$ with its nearest-neighbour spins is dealt with exactly while the interactions of the spins $\sigma_1, \ldots, \sigma_q$ with the other spins in the lattice are represented approximately by a mean field $B'$. It follows, therefore, that the system can be described by a cluster hamiltonian

$$\mathscr{H}_C = -J\sigma_0 \sum_{j=1}^{q} \sigma_j - m_0 B \sigma_0 - m_0 B_1 \sum_{i=1}^{q} \sigma_i, \tag{2.1}$$

where
$$B_1 \equiv B + B'. \tag{2.2}$$

The partition function associated with the hamiltonian (2.1) is

$$Z_C = \mu^{-\frac{1}{2}}(z^{-\frac{1}{2}}\mu_1^{-\frac{1}{2}} + z^{\frac{1}{2}}\mu_1^{\frac{1}{2}})^q + \mu^{\frac{1}{2}}(z^{-\frac{1}{2}}\mu_1^{\frac{1}{2}} + z^{\frac{1}{2}}\mu_1^{-\frac{1}{2}})^q, \tag{2.3}$$

where
$$z = u^{\frac{1}{2}} = \exp(-2J\beta), \tag{2.4}$$

$$\mu_1 = \exp(-2m_0 B_1 \beta). \tag{2.5}$$

The mean field $B'$ is determined by imposing the consistency condition

$$\langle \sigma_0 \rangle \equiv \langle \sigma_1 \rangle = \ldots = \langle \sigma_q \rangle, \tag{2.6}$$

where
$$\langle \sigma_j \rangle = Z_C^{-1} \sum_{\sigma_0 = \pm 1} \sum_{\sigma_1 = \pm 1} \ldots \sum_{\sigma_q = \pm 1} \sigma_j \exp(-\beta \mathscr{H}_C), \tag{2.7}$$

and $j = 0, 1, \ldots, q$. If (2.1) is substituted in the thermal average (2.7) it is found that the consistency condition (2.6) can be written in the form

$$\mu(\partial Z_C/\partial \mu) = (\mu_1/q)\,(\partial Z_C/\partial \mu_1). \tag{2.8}$$

The formula (2.3) is now used to evaluate (2.8). After some simplification we obtain the following implicit equation for the mean field parameter $\mu_1$:

$$\mu = \mu_1 [(1 + z\mu_1)/(z + \mu_1)]^{q-1}. \tag{2.9}$$

In a similar manner we can derive an expression for the magnetization per spin $m$ by using the equations

$$m/m_0 = \langle \sigma_0 \rangle = -2\mu Z_C^{-1}(\partial Z_C/\partial\mu) \tag{2.10}$$

and (2.9). The final result is

$$m/m_0 = (1-\mu_1^2)/(1+2z\mu_1+\mu_1^2). \tag{2.11}$$

To expand $m/m_0$ in powers of $\mu$ we first write equations (2.9) and (2.11) in the alternative forms

$$\mu = \xi[\phi(q,u,\xi)]^{-1}, \tag{2.12}$$

and

$$m/m_0 = \psi(u,\xi), \tag{2.13}$$

respectively, where

$$\phi(q,u,\xi) = u^{\frac{1}{2}q-1}[(1+\xi)/(1+u\xi)]^{q-1}, \tag{2.14}$$

$$\psi(u,\xi) = (1-u\xi^2)/(1+2u\xi+u\xi^2), \tag{2.15}$$

and $\xi = \mu_1/z$. Next the physically acceptable solution $\xi = \xi(q,u,\mu)$ of the implicit equation (2.12) is expanded as a Taylor series in powers of $\mu$ by applying the Lagrange reversion theorem (Copson 1935) to (2.12). This procedure gives

$$\xi = \sum_{n=1}^{\infty} \frac{\mu^n}{n!}[\partial_\xi^{n-1}(\phi(q,u,\xi))^n]_{\xi=0}, \tag{2.16}$$

where $\partial_\xi \equiv \partial/\partial\xi$. It is now possible, at least in principle, to obtain the high-field series for $m/m_0$ by substituting (2.16) in (2.13). Fortunately, this stage of the calculation can also be carried out in closed-form by using a generalization of the standard Lagrange reversion theorem (Copson 1935). In this manner we find that

$$\frac{m}{m_0} = 1-2u\sum_{n=1}^{\infty}\frac{\mu^n}{n!}[\partial_\xi^{n-1}\{G(u,\xi)\,(\phi(q,u,\xi))^n\}]_{\xi=0}, \tag{2.17}$$

where

$$\begin{aligned} G(u,\xi) &\equiv -\tfrac{1}{2}u^{-1}\partial_\xi\psi(u,\xi) \\ &= (1+2\xi+u\xi^2)/(1+2u\xi+u\xi^2)^2. \end{aligned} \tag{2.18}$$

The comparison of this result with (1.8) yields the basic formula

$$nL_n(u) = u^{\frac{1}{2}nq-n+1}S_{n-1}(q,u), \tag{2.19}$$

where

$$S_{n-1}(q,u) = \frac{1}{n!}\left[\partial_\xi^{n-1}\left\{G(u,\xi)\left(\frac{1+\xi}{1+u\xi}\right)^{n(q-1)}\right\}\right]_{\xi=0} \tag{2.20}$$

is a polynomial of degree $n-1$ in the variable $u$.

The formula (2.20) can be evaluated by first introducing the standard generating function (Erdélyi *et al.* 1953)

$$\frac{1-t^2}{1-2zt+t^2} = 1+2\sum_{n=1}^{\infty}T_n(z)\,t^n, \tag{2.21}$$

where $T_n(z)$ denotes a Chebyshev polynomial of the first kind. If this result is applied to (2.15) it is readily seen from (2.18) that

$$G(u,\xi) = \sum_{n=0}^{\infty} C_n(u)\,\xi^n, \tag{2.22}$$

where
$$C_n(u) = (-1)^n (n+1)\, z^{n-1} T_{n+1}(z) \tag{2.23}$$
is a polynomial of degree $n$ in the variable $u = z^2$. It follows from (2.23) and the three-term recurrence relation for $T_n(z)$ that the polynomial $C_n(u)$ satisfies the relation
$$n(n-1)\, C_n(u) + 2(n^2-1)\, uC_{n-1}(u) + n(n+1)\, uC_{n-2}(u) = 0, \tag{2.24}$$
with the initial conditions $C_0(u) = 1$ and $C_1(u) = 2(1-2u)$.

Next we consider the further generating function
$$[(1+\xi)/(1+u\xi)]^{\gamma} = \sum_{n=0}^{\infty} D_n(\gamma, u)\, \xi^n, \tag{2.25}$$
where $D_n(\gamma, u)$ is a polynomial of degree $n$ in the variable $u$. It can be shown that the polynomial $D_n(\gamma, u)$ satisfies the recurrence relation
$$nD_n(\gamma, u) + [(n-1-\gamma) + (n-1+\gamma)\, u]\, D_{n-1}(\gamma, u) + u(n-2)\, D_{n-2}(\gamma, u) = 0, \tag{2.26}$$
with the initial conditions $D_{-1}(\gamma, u) \equiv 0$ and $D_0(\gamma, u) = 1$.

The generating functions (2.22) and (2.25) enable one to express (2.20) in the simplified form
$$S_{n-1}(q, u) = \frac{1}{n} \sum_{m=0}^{n-1} C_{n-m-1}(u)\, D_m(n(q-1), u). \tag{2.27}$$
This formula has been used, in combination with the recurrence relations (2.24) and (2.26), to calculate the set of polynomials $\{S_n(q, u);\, n = 0, 1, 2, \ldots\}$. The first few polynomials are
$$S_0(q, u) = 1, \tag{2.28}$$
$$S_1(q, u) = -[(q+1)\, u - q], \tag{2.29}$$
$$S_2(q, u) = \tfrac{1}{2}[(3q^2+3q+2)\, u^2 - 6q^2 u + 3q(q-1)], \tag{2.30}$$
$$S_3(q, u) = -\tfrac{1}{3}[(2q+1)(4q^2+q+3)\, u^3 - 6q(4q^2-q+1)\, u^2 + 3q(2q-1)(4q-3)\, u - 2q(q-1)(4q-5)]. \tag{2.31}$$
An alternative closed-form expression for $S_{n-1}(q, u)$ has been given by Bessis *et al.* (1976) in terms of Jacobi polynomials and Chebyshev polynomials of the second kind.

There are several special cases of the polynomial $S_n(q, u)$ which are of interest. For example, it can be shown from (2.20) that
$$S_n(q, 1) = (-1)^n, \tag{2.32}$$
$$S_n(q, 0) = \frac{q((n+1)\, q - (n+1))!}{n!((n+1)\, q - 2n)!}. \tag{2.33}$$
It is also possible to prove that
$$S_n(0, u) = (-u)^n, \tag{2.34}$$
$$S_n(1, u) = (-1)^n z^{n-1} T_{n+1}(z), \tag{2.35}$$
where $z = u^{\frac{1}{2}}$.

For the case $q = 2$ and $\mu < 1$ the physically acceptable solution $\xi$ of the implicit equation (2.12) is given by
$$2u\xi = -(1-\mu) + [(1-\mu)^2 + 4u\mu]^{\frac{1}{2}}. \tag{2.36}$$

The substitution of (2.36) in (2.13) yields the following exact result for the one-dimensional Ising model (Domb 1960):

$$m/m_0 = (1-\mu)\,[(1-\mu)^2+4u\mu]^{-\frac{1}{2}}. \tag{2.37}$$

From this formula and the work of Bessis *et al.* (1976) and Joyce (1990) it is found that

$$S_n(2,u) = P_n^{(1,0)}(1-2u), \tag{2.38}$$

where $P_n^{(\alpha,\beta)}(x)$ denotes the Jacobi polynomial of degree $n$.

## 3. Application of the saddle-point method

The main aim in this section is to determine the asymptotic behaviour of $L_n(u)$ as $n \to \infty$ with $u$ fixed in the interval $(0, u_c)$, where

$$u_c = [(q-2)/q]^2 \tag{3.1}$$

is the critical value of the variable $u$ and $q > 2$. We begin by applying the Cauchy integral representation for the $(n-1)$th derivative of an analytic function (Copson 1935) to (2.20). This procedure enables one to write (2.19) in the alternative form

$$n^2 L_n(u) = u^{\frac{1}{2}nq-n+1} I_n(q,u), \tag{3.2}$$

where
$$I_n(q,u) = \frac{1}{2\pi i}\int_\Gamma G(u,x)\exp[nF(\sigma,u,x)]\,dx, \tag{3.3}$$

$$F(\sigma,u,x) = -\ln x+\sigma\ln(1+x)-\sigma\ln(1+ux), \tag{3.4}$$

$$\sigma = q-1, \tag{3.5}$$

and $\Gamma$ denotes a closed contour in the $x$-plane which does not enclose any of the singularities of the integrand in (3.3) except the pole of order $n$ at $x = 0$. Bessis *et al.* (1976) have also derived an alternative contour integral representation for $L_n(u)$. The connection between the two representations can be established by first applying the transformation $x = (1-\xi)/(\xi-u)$ to (3.3), and then integrating by parts. It should be noted, however, that there appears to be an error in the second part of the final formula (6.22) given by Bessis *et al.*

From (3.4) and the condition $(\partial F/\partial x) = 0$ it is readily found that $F(\sigma,u,x)$ has two saddle points $x_\pm(\sigma,u)$ which are the solutions of the quadratic equation

$$ux^2+[(\sigma+1)\,u-(\sigma-1)]\,x+1 = 0. \tag{3.6}$$

It is convenient to apply the transformation

$$x = (\sigma-1)^{-1}(1+\sigma\bar{x}) \tag{3.7}$$

to (3.6). In this manner we obtain the modified saddle-point equation

$$\sigma\bar{x}^2+[(\sigma^2+1)-(\sigma-1)^2u^{-1}]\,\bar{x}+\sigma = 0. \tag{3.8}$$

When $0 < u \leqslant u_c$ we can use the parametric representation

$$u = (\sigma-1)^2/[(\sigma^2+1)+2\sigma\cosh\vartheta], \tag{3.9}$$

where $0 \leqslant \vartheta < \infty$, to write (3.8) in the simplified form

$$\bar{x}^2-2\,(\cosh\vartheta)\,\bar{x}+1 = 0. \tag{3.10}$$

From this result it is seen that

$$x_{\pm} = (\sigma-1)^{-1}(1+\sigma\,\mathrm{e}^{\pm\vartheta}), \tag{3.11}$$

with $0 \leqslant \vartheta < \infty$. For the case $u_c < u \leqslant 1$ the parametric representation (3.9) is replaced by

$$u = (\sigma-1)^2/[(\sigma^2+1)+2\sigma\cos\theta], \tag{3.12}$$

where $0 < \theta \leqslant \pi$. The substitution of (3.12) in (3.8) gives the saddle points

$$x_{\pm} = (\sigma-1)^{-1}(1+\sigma\,\mathrm{e}^{\pm\mathrm{i}\theta}), \tag{3.13}$$

with $0 < \theta \leqslant \pi$. It should be noted that the saddle points $x_{\pm}$ are coincident when $u = u_c$ and $u = 1$.

The asymptotic behaviour of the integral (3.3) as $n \to \infty$, with $u$ fixed in the interval $(0, u_c)$, may now be determined by allowing the contour $\Gamma$ to pass through the saddle point

$$x_- = x_-(\sigma, u) = (\sigma-1)^{-1}(1+\sigma\,\mathrm{e}^{-\vartheta}), \tag{3.14}$$

which is closest to the origin in the $x$-plane. Next we expand the functions $F(\sigma,u,x)$ and $G(u,x)$ as Taylor series about $x_-$ and follow the standard saddle-point method (Dingle 1973). To leading order this procedure gives

$$I_n(q,u) \sim \frac{G_0\,\mathrm{e}^{nF_0}}{2\pi}\int_{-\infty}^{+\infty}\exp\left(-\tfrac{1}{2}nF_2\,y^2\right)\mathrm{d}y, \tag{3.15}$$

as $n \to \infty$, where

$$G_0 \equiv G(u,x_-) = \frac{(\sigma+1)\,(\sigma+\mathrm{e}^{-\vartheta})}{\sigma(\sigma-1)\,(1+\mathrm{e}^{-\vartheta})^2}, \tag{3.16}$$

$$F_0 \equiv F(\sigma,u,x_-) = \ln\left[\frac{(\sigma+\mathrm{e}^{-\vartheta})^{\sigma}}{(\sigma-1)^{\sigma-1}(1+\sigma\,\mathrm{e}^{-\vartheta})}\right], \tag{3.17}$$

$$F_2 \equiv [\partial_x^2 F(\sigma,u,x)]_{x=x_-} = \frac{(\sigma-1)^3(1-\mathrm{e}^{-\vartheta})}{\sigma(1+\mathrm{e}^{-\vartheta})\,(1+\sigma\,\mathrm{e}^{-\vartheta})^2}, \tag{3.18}$$

$0 < \vartheta < \infty$ and $\sigma > 1$. These results and (3.2) lead to the asymptotic representation

$$L_n(u) \sim a(\sigma,\vartheta)\,n^{-\frac{5}{2}}\exp\left[-\lambda(\sigma,\vartheta)\,n\right], \tag{3.19}$$

as $n \to \infty$, where

$$a(\sigma,\vartheta) = \tfrac{1}{2}(\sigma+1)\left[2\pi\sigma(\sigma-1)\,(1+\cosh\vartheta)\sinh\vartheta\right]^{-\frac{1}{2}}, \tag{3.20}$$

$$\lambda(\sigma,\vartheta) = \tfrac{1}{2}\ln\left[\mathrm{e}^{(\sigma-1)\vartheta}((\sigma+\mathrm{e}^{\vartheta})/(1+\sigma\,\mathrm{e}^{\vartheta}))^{\sigma+1}\right], \tag{3.21}$$

and $0 < \vartheta < \infty$, and $\sigma > 1$.

Higher-order asymptotic representations for $L_n(u)$ can also be derived by substituting the values of the derivatives

$$F_n \equiv [\partial_x^n F(\sigma,u,x)]_{x=x_-}, \tag{3.22}$$

$$G_n \equiv [\partial_x^n G(u,x)]_{x=x_-}, \tag{3.23}$$

where $n = 0, 1, 2, \ldots$, in the extensive set of formulae given by Dingle (1973). This procedure involves a large amount of complicated algebra which was carried out using the REDUCE computer algebra program (Hearn 1968; Rayna 1987). The final result is

$$L_n(u) \sim a(\sigma,\vartheta)\,n^{-\frac{5}{2}}\exp\left[-\lambda(\sigma,\vartheta)\,n\right]\sum_{r=0}^{\infty} V_r(\sigma,\vartheta)\,n^{-r}, \tag{3.24}$$

as $n \to \infty$, where

$$V_r(\sigma, \vartheta) = [24(\sigma-1)(1-\cosh\vartheta)\sinh\vartheta]^{-r}\, W_r(\sigma, \vartheta), \tag{3.25}$$

$$W_0(\sigma, \vartheta) = 1, \tag{3.26}$$

$$W_1(\sigma, \vartheta) = [2(s_1+35)\cosh^2\vartheta - 2(s_1+65)\cosh\vartheta + 5(s_1+14)], \tag{3.27}$$

$$\begin{aligned} W_2(\sigma, \vartheta) = \tfrac{1}{2}[&4(s_2+70s_1+1947)\cosh^4\vartheta - 8(s_2+532s_1+5517)\cosh^3\vartheta \\ &+12(2s_2+917s_1+7149)\cosh^2\vartheta \\ &+4(67s_2-2051s_1-16\,656)\cosh\vartheta + (97s_2+2716s_1+19\,494)], \end{aligned} \tag{3.28}$$

$$\begin{aligned} W_3(\sigma, \vartheta) = \tfrac{1}{30}[&8(-139s_3+957s_2+28\,758s_1+749\,089)\cosh^6\vartheta \\ &+24(139s_3-5427s_2-578\,538s_1-3\,530\,179)\cosh^5\vartheta \\ &+12(-109s_3+36\,582s_2+5\,455\,428s_1+27\,684\,664)\cosh^4\vartheta \\ &+4(992s_3-126\,186s_2-31\,779\,024s_1-150\,105\,722)\cosh^3\vartheta \\ &+6(25\,807s_3+413\,379s_2+22\,250\,511s_1+96\,657\,758)\cosh^2\vartheta \\ &+6(30\,691s_3-393s_2-10\,948\,407s_1-46\,737\,286)\cosh\vartheta \\ &+(81\,553s_3+262\,986s_2+13\,874\,559s_1+55\,924\,172)], \end{aligned} \tag{3.29}$$

$$s_n = \sigma^n + \sigma^{-n}, \tag{3.30}$$

and $0 < \vartheta < \infty$.

The basic asymptotic expansion (3.24) for $L_n(u)$ is only applicable for $0 < u < u_c$. Furthermore, the expansion (3.24) clearly breaks down as $n \to \infty$ and $\vartheta \to 0+$ with $n\vartheta^3$ small. However, we shall find in the following section that (3.24) plays a crucial role in the derivation of a uniform asymptotic expansion for $L_n(u)$ which has a wider range of validity.

## 4. Uniform asymptotic expansion for $L_n(u)$

In the neighbourhood of $\vartheta = 0+$ the two saddle points (3.11) are nearly coincident and the asymptotic expansion (3.24) is not uniformly valid. Under these circumstances, we can follow an alternative procedure which was developed by Chester *et al.* (1957).

In this method the implicit cubic transformation

$$F(\sigma, u, x) = \tfrac{1}{3}\omega^3 - \zeta(\sigma, \vartheta)\,\omega + A(\sigma, \vartheta) \tag{4.1}$$

is used to introduce a new complex variable $\omega$. The parameters $\zeta(\sigma, \vartheta)$ and $A(\sigma, \vartheta)$ in (4.1) are defined to be

$$\zeta(\sigma, \vartheta) \equiv (\tfrac{3}{4})^{\frac{2}{3}}[F(\sigma, u, x_+) - F(\sigma, u, x_-)]^{\frac{2}{3}}, \tag{4.2}$$

$$A(\sigma, \vartheta) \equiv \tfrac{1}{2}[F(\sigma, u, x_+) + F(\sigma, u, x_-)], \tag{4.3}$$

where $x_\pm$ denote the saddle points (3.11). From these expressions one finds that

$$\tfrac{2}{3}[\zeta(\sigma, \vartheta)]^{\frac{3}{2}} = \lambda(\sigma, \vartheta), \tag{4.4}$$

$$A(\sigma, \vartheta) = -\tfrac{1}{2}(\sigma-1)\ln u, \tag{4.5}$$

where $\lambda(\sigma, \vartheta)$ and $u = u(\sigma, \vartheta)$ are defined in equations (3.21) and (3.9) respectively. Chester *et al.* (1957) have proved that just one branch of the transformation (4.1) gives a mapping $x \leftrightarrow \omega$ which is uniformly regular for sufficiently small $\omega$ and $\vartheta$. On this regular branch $x = x(\omega, \vartheta)$ the saddle points $x = x_+$ and $x = x_-$ correspond to the points $\omega = -\zeta^{\frac{1}{2}}$ and $\omega = +\zeta^{\frac{1}{2}}$ respectively.

Next we consider the uniform expansion

$$G(u, x)\frac{\mathrm{d}x}{\mathrm{d}\omega} = \sum_{m=0}^{\infty} p_m(\sigma, \vartheta)\,(\omega^2-\zeta)^m + \sum_{m=0}^{\infty} q_m(\sigma, \vartheta)\,\omega(\omega^2-\zeta)^m, \tag{4.6}$$

which is convergent for sufficiently small $\omega$ and $\vartheta$. The coefficients $p_m(\sigma, \vartheta)$ and $q_m(\sigma, \vartheta)$ in this expansion are determined successively by evaluating the $m$th derivative of equation (4.6) with respect to $\omega$ at the two corresponding sets of points $\{x = x_+, \omega = -\zeta^{\frac{1}{2}}\}$ and $\{x = x_-, \omega = +\zeta^{\frac{1}{2}}\}$. The derivatives of $x = x(\omega, \vartheta)$ with respect to $\omega$ which are required in these calculations are found by repeated differentiation of the transformation (4.1). To leading order this complicated procedure yields

$$p_0(\sigma, \vartheta) = -\frac{2^{\frac{1}{2}}(\sigma+1)\,(\sigma+\mathrm{e}^{\vartheta})\,(1+\sigma\,\mathrm{e}^{\vartheta})\,\zeta^{\frac{1}{4}}}{[\sigma(\sigma-1)^5(\mathrm{e}^{\vartheta}+1)^3(\mathrm{e}^{\vartheta}-1)]^{\frac{1}{2}}}, \tag{4.7}$$

and $q_0(\sigma, \vartheta) = 0$, where $\zeta = \zeta(\sigma, \vartheta)$ is defined in (4.4). Higher-order calculations indicate that the coefficients $q_m(\sigma, \vartheta)$ are identically equal to zero for all $m = 1, 2, \ldots$. (A general proof of this last result has not yet been established.)

We now substitute (4.1) and (4.6) in the transformed integral (3.3) and apply (3.2), (4.5), (4.7) and the identity $q_m(\sigma, \vartheta) \equiv 0$. In this manner we obtain

$$n^2 L_n(u) \sim 2\pi^{\frac{1}{2}} a(\sigma, \vartheta)\,\zeta^{\frac{1}{4}} \sum_{m=0}^{\infty} \frac{p_m(\sigma, \vartheta)}{p_0(\sigma, \vartheta)} J_m(\zeta, n, C_1), \tag{4.8}$$

where $a(\sigma, \vartheta)$ is defined in (3.20),

$$J_m(\zeta, n, C_1) = \frac{1}{2\pi \mathrm{i}} \int_{C_1} (\omega^2-\zeta)^m \exp\left[n(\tfrac{1}{3}\omega^3 - \zeta\omega)\right] \mathrm{d}\omega, \tag{4.9}$$

and the contour $C_1$ is taken from $\infty \exp(-\frac{1}{3}\pi \mathrm{i})$ to $\infty \exp(+\frac{1}{3}\pi \mathrm{i})$. Chester *et al.* (1957) have shown that the expansion (4.8) can be written in the alternative form

$$L_n(u) \sim 2\pi^{\frac{1}{2}} a(\sigma, \vartheta)\,\zeta^{\frac{1}{4}} n^{-\frac{7}{3}} \left[\mathrm{Ai}\,(\zeta n^{\frac{2}{3}}) \sum_{m=0}^{\infty} A_m(\sigma, \vartheta)\, n^{-2m} + n^{-\frac{4}{3}} \mathrm{Ai}'\,(\zeta n^{\frac{2}{3}}) \sum_{m=0}^{\infty} B_m(\sigma, \vartheta)\, n^{-2m}\right], \tag{4.10}$$

as $n \to \infty$, where $\mathrm{Ai}\,(z)$ denotes the Airy function and $A_0(\sigma, \vartheta) \equiv 1$.

In order to determine the coefficients $A_m(\sigma, \vartheta)$ and $B_m(\sigma, \vartheta)$ in the basic uniform asymptotic expansion (4.10) it is clear that we must first obtain formulae for the higher-order coefficients $p_1(\sigma, \vartheta), p_2(\sigma, \vartheta), \ldots$. The derivation of these formulae from (4.6) involves a large amount of very complicated algebra. Fortunately, there is an alternative procedure which is much simpler!

In this method the Airy function and its first derivative in (4.10) are replaced by the following standard asymptotic representations:

$$\mathrm{Ai}\,(\zeta n^{\frac{2}{3}}) \sim \tfrac{1}{2}\pi^{-\frac{1}{2}} \zeta^{-\frac{1}{4}} n^{-\frac{1}{6}} \mathrm{e}^{-\lambda n} \sum_{k=0}^{\infty} (-1)^k c_k (n\zeta^{\frac{3}{2}})^{-k}, \tag{4.11}$$

$$\mathrm{Ai}'\,(\zeta n^{\frac{2}{3}}) \sim -\tfrac{1}{2}\pi^{-\frac{1}{2}} \zeta^{\frac{1}{4}} n^{\frac{1}{6}} \mathrm{e}^{-\lambda n} \sum_{k=0}^{\infty} (-1)^k d_k (n\zeta^{\frac{3}{2}})^{-k}, \tag{4.12}$$

as $n \to \infty$ with $0 < \vartheta < \infty$, where $\lambda = \frac{2}{3}\zeta^{\frac{3}{2}}$,

$$c_k = \frac{(2k+1)(2k+3)\dots(6k-1)}{(144)^k\, k!}, \tag{4.13}$$

$$d_k = -[(6k+1)/(6k-1)]\, c_k, \tag{4.14}$$

for $k = 1, 2, \dots$, with $c_0 = d_0 = 1$. A comparison of the resulting asymptotic expansion with the equivalent saddle-point expansion (3.24) enables one to derive the required formulae for the coefficients $A_m(\sigma, \vartheta)$ and $B_m(\sigma, \vartheta)$. In particular, it is found that

$$A_m(\sigma, \vartheta) = \sum_{k=0}^{2m} d_k\, \zeta^{-3k/2}\, V_{2m-k}(\sigma, \vartheta), \tag{4.15}$$

$$\zeta^{\frac{1}{2}} B_m(\sigma, \vartheta) = -\sum_{k=0}^{2m+1} c_k\, \zeta^{-3k/2}\, V_{2m+1-k}(\sigma, \vartheta), \tag{4.16}$$

where $m = 0, 1, 2, \dots$, and $\zeta = \zeta(\sigma, \vartheta)$ is defined in (4.4). From these general results and (3.25) we readily obtain

$$A_0(\sigma, \vartheta) = 1, \tag{4.17}$$

$$B_0(\sigma, \vartheta) = \tfrac{1}{48}\zeta^{-2}[-5 + 2\delta_0\, W_1(\sigma, \vartheta)], \tag{4.18}$$

$$A_1(\sigma, \vartheta) = \tfrac{1}{4608}\zeta^{-3}[-455 + 28\delta_0\, W_1(\sigma, \vartheta) + 8\delta_0^2\, W_2(\sigma, \vartheta)], \tag{4.19}$$

$$B_1(\sigma, \vartheta) = \tfrac{1}{663552}\zeta^{-5}[-85085 + 2310\delta_0\, W_1(\sigma, \vartheta) - 120\delta_0^2\, W_2(\sigma, \vartheta) + 48\delta_0^3\, W_3(\sigma, \vartheta)], \tag{4.20}$$

where
$$\delta_0 = \delta_0(\sigma, \vartheta) \equiv \frac{[\zeta(\sigma, \vartheta)]^{\frac{3}{2}}}{(\sigma - 1)(\cosh\vartheta - 1)\sinh\vartheta}. \tag{4.21}$$

It follows from equations (3.21) and (4.4) that the function $\zeta(\sigma, \vartheta)$ can be expanded about $\vartheta = 0$ in the form

$$\zeta(\sigma, \vartheta) = [\tfrac{1}{4}\sigma(\sigma^2 - 1)]^{\frac{2}{3}} \sum_{k=0}^{\infty} \zeta_k(\sigma) \left(\frac{\vartheta}{1+\sigma}\right)^{2k+2}, \tag{4.22}$$

where the coefficients $\zeta_k(\sigma)$ are symmetric polynomials of degree $k$ in the variable $\sigma$. Formulae for $\zeta_k(\sigma)$ are listed in Appendix 1 for $k \leqslant 7$. The direct substitution of (4.22) in equations (4.15) and (4.16) indicates that the coefficients $A_m(\sigma, \vartheta)$ and $B_m(\sigma, \vartheta)$ have poles of order $6m$ and $6m+4$ respectively at $\vartheta = 0$. However, if the Laurent series about $\vartheta = 0$ are derived for these coefficients we find that the singular parts of the series cancel exactly and we can write

$$A_m(\sigma, \vartheta) = \left[\frac{4}{\sigma(\sigma^2 - 1)}\right]^{2m} \sum_{k=0}^{\infty} A_k^{(m)}(\sigma) \left(\frac{\vartheta}{1+\sigma}\right)^{2k}, \tag{4.23}$$

$$B_m(\sigma, \vartheta) = \left[\frac{4}{\sigma(\sigma^2 - 1)}\right]^{2m+\frac{4}{3}} \sum_{k=0}^{\infty} B_k^{(m)}(\sigma) \left(\frac{\vartheta}{1+\sigma}\right)^{2k}, \tag{4.24}$$

where $m = 0, 1, \dots$, with $A_0(\sigma, \vartheta) \equiv 1$. Expressions for the first few coefficients $A_k^{(1)}(\sigma)$, $B_k^{(0)}(\sigma)$ and $B_k^{(1)}(\sigma)$ are given in Appendixes 2, 3 and 4 respectively.

We are now able to use the Taylor series (4.22)–(4.24) to evaluate the basic uniform asymptotic expansion (4.10) in the limit $\vartheta \to 0+$. In this manner we obtain the asymptotic representation

$$L_n(u_c) \sim \frac{1}{2^{\frac{4}{3}}}\left[\frac{(\sigma+1)^2}{\sigma(\sigma-1)}\right]^{\frac{1}{3}} n^{-\frac{7}{3}}\left[\mathrm{Ai}\,(0) \sum_{m=0}^{\infty} A_0^{(m)}(\sigma)\left\{\frac{4}{\sigma(\sigma^2-1)\,n}\right\}^{2m} + \mathrm{Ai}'\,(0) \sum_{m=0}^{\infty} B_0^{(m)}(\sigma)\left\{\frac{4}{\sigma(\sigma^2-1)\,n}\right\}^{2m+\frac{4}{3}}\right], \quad (4.25)$$

as $n \to \infty$, where $A_0^{(0)}(\sigma) \equiv 1$,

$$\mathrm{Ai}\,(0) = 3^{-\frac{1}{6}}(2\pi)^{-1}\Gamma(\tfrac{1}{3}), \quad (4.26)$$

$$-\mathrm{Ai}'\,(0) = 3^{\frac{1}{6}}(2\pi)^{-1}\Gamma(\tfrac{2}{3}), \quad (4.27)$$

and $\Gamma(x)$ denotes the gamma function. It should be noted that the special case (4.25) could also have been derived by using a modification of the saddle-point method described in §3 with $F_1 = F_2 = 0$ and $F_3 \neq 0$ (see Dingle 1973, p. 136).

It is possible to write the basic asymptotic expansion (4.10) in the alternative form

$$L_n(u) \sim 2\pi^{\frac{1}{2}} a(\sigma,\vartheta)\,\zeta^{\frac{1}{4}} n^{-\frac{7}{3}}\left[1 + \sum_{m=1}^{\infty} E_m(\sigma,\vartheta)\,n^{-2m}\right]\mathrm{Ai}\,(\zeta n^{\frac{2}{3}} + \epsilon), \quad (4.28)$$

where

$$\epsilon = \epsilon(n,\sigma,\vartheta) = n^{-\frac{4}{3}} \sum_{m=0}^{\infty} H_m(\sigma,\vartheta)\,n^{-2m}. \quad (4.29)$$

The coefficients $E_m(\sigma,\vartheta)$ and $H_m(\sigma,\vartheta)$ may be related to the coefficients $A_m(\sigma,\vartheta)$ and $B_m(\sigma,\vartheta)$ by first developing the Airy function in (4.28) as a Taylor series in powers of $\epsilon$. Next we replace $\epsilon$ in this series by (4.29) and use the standard differential equation

$$\mathrm{Ai}''\,(z) = z\,\mathrm{Ai}\,(z), \quad (4.30)$$

to express the higher-order derivatives of the Airy function in terms of $\mathrm{Ai}\,(z)$ and $\mathrm{Ai}'\,(z)$. Finally, the resulting asymptotic expansion for $L_n(u)$ is compared with (4.10). In this manner we obtain

$$H_0(\sigma,\vartheta) = B_0(\sigma,\vartheta), \quad (4.31)$$

$$E_1(\sigma,\vartheta) = A_1(\sigma,\vartheta) - \tfrac{1}{2}[B_0(\sigma,\vartheta)]^2\,\zeta, \quad (4.32)$$

$$H_1(\sigma,\vartheta) = B_1(\sigma,\vartheta) - B_0(\sigma,\vartheta)\,A_1(\sigma,\vartheta) + \tfrac{1}{3}[B_0(\sigma,\vartheta)]^3\zeta, \quad (4.33)$$

where $\zeta = \zeta(\sigma,\vartheta)$ and $\vartheta > 0$.

We see from equations (4.22)–(4.24) that the coefficients $E_m(\sigma,\vartheta)$ and $H_m(\sigma,\vartheta)$ have Taylor series representations about $\vartheta = 0$. In particular, it is found that the coefficient $H_m(\sigma,\vartheta)$ can be expanded in the form

$$H_m(\sigma,\vartheta) = \left[\frac{4}{\sigma(\sigma^2-1)}\right]^{2m+\frac{4}{3}} \sum_{k=0}^{\infty} H_k^{(m)}(\sigma)\left(\frac{\vartheta}{1+\sigma}\right)^{2k}, \quad (4.34)$$

where $m = 0, 1, \ldots$, and

$$H_k^{(0)}(\sigma) \equiv B_k^{(0)}(\sigma). \quad (4.35)$$

We can determine the first few coefficients $\{H_k^{(1)}(\sigma);\, k = 0, 1, \ldots\}$ using the formulae listed in Appendixes 1–4. The final results are given in Appendix 5. In §6 we shall find that the alternative asymptotic formula (4.28) is especially useful for analysing the asymptotic properties of the zeros of $L_n(u)$.

## 5. Asymptotic behaviour of $L_n(u)$ for $u_c < u < 1$

It is seen from the parametric equations (3.9) and (3.12) that the asymptotic properties of $L_n(u)$ for $u_c < u < 1$ can be established by applying the transformation $\vartheta = \mathrm{i}\theta$, with $0 < \theta < \pi$, to the results given in the previous sections. We begin this procedure by using (3.21) and (4.4) to obtain the transformation relations

$$\zeta(\sigma, \mathrm{i}\theta) = -\bar{\zeta}(\sigma, \theta), \tag{5.1}$$

$$[\zeta(\sigma, \mathrm{i}\theta)]^{\frac{3}{2}} = -\mathrm{i}[\bar{\zeta}(\sigma, \theta)]^{\frac{3}{2}}, \tag{5.2}$$

where

$$\bar{\zeta}(\sigma, \theta) = (\tfrac{3}{4})^{\frac{2}{3}}\left[-(\sigma-1)\,\theta+(\sigma+1)\arccos\left\{\frac{2\sigma+(\sigma^2+1)\cos\theta}{(\sigma^2+1)+2\sigma\cos\theta}\right\}\right]^{\frac{2}{3}}, \tag{5.3}$$

and $0 \leqslant \theta < \pi$. It should be noted that a formula similar to (5.3) can also be derived for the function $\zeta(\sigma, \vartheta)$. In particular, we find that

$$\zeta(\sigma, \vartheta) = (\tfrac{3}{4})^{\frac{2}{3}}\left[+(\sigma-1)\,\vartheta-(\sigma+1)\,\mathrm{arccosh}\left\{\frac{2\sigma+(\sigma^2+1)\cosh\vartheta}{(\sigma^2+1)+2\sigma\cosh\vartheta}\right\}\right]^{\frac{2}{3}}, \tag{5.4}$$

where $\vartheta \geqslant 0$. Next we use equations (5.2) and (4.21) to obtain the further relation

$$\delta_0(\sigma, \mathrm{i}\theta) = \bar{\delta}_0(\sigma, \theta), \tag{5.5}$$

where
$$\bar{\delta}_0(\sigma, \theta) = \frac{[\bar{\zeta}(\sigma, \theta)]^{\frac{3}{2}}}{(\sigma-1)\,(1-\cos\theta)\sin\theta}, \tag{5.6}$$

and $0 < \theta < \pi$.

We are now able to apply the transformation $\vartheta = \mathrm{i}\theta$ to the basic uniform asymptotic expansion (4.10). This procedure yields

$$L_n(u) \sim 2\pi^{\frac{1}{2}}\bar{a}(\sigma, \theta)\,(\bar{\zeta})^{\frac{1}{4}}\,n^{-\frac{7}{3}}\left[\mathrm{Ai}\,(-\bar{\zeta}n^{\frac{2}{3}})\sum_{m=0}^{\infty}\bar{A}_m(\sigma, \theta)\,n^{-2m} + n^{-\frac{4}{3}}\,\mathrm{Ai}'\,(-\bar{\zeta}n^{\frac{2}{3}})\sum_{m=0}^{\infty}\bar{B}_m(\sigma, \theta)\,n^{-2m}\right], \tag{5.7}$$

as $n \to \infty$, where $0 < \theta < \pi$,

$$\bar{A}_m(\sigma, \theta) = A_m(\sigma, \mathrm{i}\theta), \tag{5.8}$$

$$\bar{B}_m(\sigma, \theta) = B_m(\sigma, \mathrm{i}\theta), \tag{5.9}$$

$$\bar{a}(\sigma, \theta) = \tfrac{1}{2}(\sigma+1)\,[2\pi\sigma(\sigma-1)\,(1+\cos\theta)\sin\theta]^{-\frac{1}{2}}, \tag{5.10}$$

and $\bar{\zeta} = \bar{\zeta}(\sigma, \theta)$. Explicit formulae for the coefficients $\bar{A}_m(\sigma, \theta)$ and $\bar{B}_m(\sigma, \theta)$ are given below for $m = 0$ and $1$:

$$\bar{A}_0(\sigma, \theta) = 1, \tag{5.11}$$

$$\bar{B}_0(\sigma, \theta) = \tfrac{1}{48}\bar{\zeta}^{-2}[-5+2\bar{\delta}_0\,\bar{W}_1(\sigma, \theta)], \tag{5.12}$$

$$\bar{A}_1(\sigma, \theta) = \tfrac{1}{4608}\bar{\zeta}^{-3}[455-28\bar{\delta}_0\,\bar{W}_1(\sigma, \theta)-8(\bar{\delta}_0)^2\bar{W}_2(\sigma, \theta)], \tag{5.13}$$

$$\bar{B}_1(\sigma, \theta) = \tfrac{1}{663\,552}\bar{\zeta}^{-5}[85085-2310\bar{\delta}_0\,\bar{W}_1(\sigma, \theta)+120(\bar{\delta}_0)^2\bar{W}_2(\sigma, \theta)-48(\bar{\delta}_0)^3\bar{W}_3(\sigma, \theta)], \tag{5.14}$$

where
$$\bar{W}_r(\sigma, \theta) \equiv W_r(\sigma, \mathrm{i}\theta), \tag{5.15}$$

and $\bar{\delta}_0 = \bar{\delta}_0(\sigma, \theta)$. The coefficients $\{\bar{W}_r(\sigma, \theta); r = 1, 2, 3\}$ are readily obtained from equations (3.27)–(3.29) respectively by replacing $\cosh\vartheta$ with $\cos\theta$.

When $\theta \to 0+$ the uniform asymptotic expansion (5.7) reduces to the critical case (4.25). In the critical region $|u-u_c| \ll 1$ the dominant asymptotic behaviour of $L_n(u)$ can be determined by applying the Taylor series (4.22) to the leading-order term in (5.7). We find that

$$L_n(u) \sim 2^{-\frac{4}{3}}\sigma^{-\frac{1}{3}}(\sigma-1)^{-\frac{1}{3}}(\sigma+1)^{\frac{2}{3}}n^{-\frac{7}{3}}\mathrm{Ai}\,[-2^{-\frac{4}{3}}\sigma^{-\frac{1}{3}}(\sigma^2-1)^{\frac{2}{3}}((u/u_c)-1)\,n^{\frac{2}{3}}], \quad (5.16)$$

as $n \to \infty$ and $u \to u_c \pm$. This result is in agreement with the general scaling-law prediction (Gaunt 1978)

$$L_n(u) \sim n^{-2-(1/\delta)} f(y), \quad (5.17)$$

where $f(y)$ is a function of the variable

$$y = ((u/u_c)-1)\,n^{1/\Delta}, \quad (5.18)$$

and $\delta, \Delta$ are standard critical exponents. For the Bethe approximation it is well known that $\delta = 3$ and $\Delta = \frac{3}{2}$.

The uniform asymptotic expansion (5.7) breaks down in the limit $\theta \to \pi-$ because the saddle points $x_\pm$, and the poles and zeros of the rational integrand function $G(u,x)$ in (3.3) all coalesce when $\theta = \pi$, $u = 1$. For this more complicated case it has been shown by one of us (Joyce, unpublished work) that the dominant asymptotic representation for $L_n(u)$ can be written in the form

$$L_n(u) \sim (-1)^{n-1} n^{-1} J(\sigma, \eta), \quad (5.19)$$

as $n \to \infty$ and $u \to 1\pm$, where

$$J(\sigma,\eta) = \frac{4}{\pi}\sigma(\sigma+1)\int_0^{\pi/2} \frac{\cos(2\sigma^{\frac{1}{2}}\eta\cos\theta)\sin^2\theta}{[(\sigma+1)^2\sin^2\theta+(\sigma-1)^2\cos^2\theta]}\,\mathrm{d}\theta, \quad (5.20)$$

$$\eta = (1-u)^{\frac{1}{2}} n, \quad (5.21)$$

and $\sigma > 0$. Currently, we are investigating the possibility of replacing (5.19) with a complete asymptotic expansion which is uniformly valid in the neighbourhood of $u = 1$. One approach to this problem is to apply the generalized saddle-point methods developed by Bleistein (1967), and Bleistein & Handelsman (1986) to the contour integral (3.3).

When $\sigma = 1$, $q = 2$ the formula (5.20) reduces to

$$J(1,\eta) = \frac{2}{\pi}\int_0^{\pi/2} \cos(2\eta\cos\theta)\,\mathrm{d}\theta \quad (5.22)$$

$$= J_0(2\eta), \quad (5.23)$$

where $J_0(z)$ denotes a Bessel function of the first kind. This result is in agreement with the earlier work of Joyce (1990) on the one-dimensional Ising model. For general values of $\sigma > 1$ the function $J(\sigma,\eta)$ can be represented by the Taylor series

$$J(\sigma,\eta) = \sum_{k=0}^{\infty} j_k(\sigma)\frac{(-\eta^2)^k}{(2k)!}, \quad (5.24)$$

where $|\eta| < \infty$, and the coefficient $j_k(\sigma)$ satisfies the recurrence relation

$$(k+1)\,j_{k+1}(\sigma) - [(\sigma^2+6\sigma+1)\,k + (\sigma^2+1)]\,j_k(\sigma) + 2\sigma(\sigma+1)^2(2k-1)\,j_{k-1}(\sigma) = 0, \quad (5.25)$$

with the initial conditions $j_0(\sigma) = 1$ and $j_1(\sigma) = \sigma+1$.

When $\bar{\zeta}n^{\frac{2}{3}} \gg 1$ we can replace the Airy functions in (5.7) with their standard asymptotic representations (Antosiewicz 1965). In this manner we obtain the non-uniform asymptotic expansion

$$L_n(u) \sim 2\bar{a}(\sigma,\theta)\, n^{-\frac{5}{2}}\left[\sin\left(\tfrac{2}{3}n(\bar{\zeta})^{\frac{3}{2}}+\tfrac{1}{4}\pi\right)\sum_{r=0}^{\infty}\frac{(-1)^r\,\bar{W}_{2r}(\sigma,\theta)}{(24(\sigma-1)(1-\cos\theta)\sin\theta)^{2r}n^{2r}}\right.$$
$$\left.-\cos\left(\tfrac{2}{3}n(\bar{\zeta})^{\frac{3}{2}}+\tfrac{1}{4}\pi\right)\sum_{r=0}^{\infty}\frac{(-1)^r\,\bar{W}_{2r+1}(\sigma,\theta)}{(24(\sigma-1)(1-\cos\theta)\sin\theta)^{2r+1}n^{2r+1}}\right], \quad (5.26)$$

as $n\to\infty$, with $\theta$ fixed in the interval $0<\theta<\pi$. This result could also have been established by applying the ordinary saddle-point method (Dingle 1973) to the two complex conjugate points $x_{\pm}$ which are defined in (3.13). It is interesting to note that the dominant asymptotic behaviour of the scaling function $J(\sigma,\eta)$ as $\eta\to\infty$ can be determined by taking the limit $\theta\to\pi-$ in the leading-order term of the expansion (5.26). We find that

$$J(\sigma,\eta) \sim \pi^{-\frac{1}{2}}\sigma^{\frac{1}{4}}(\sigma+1)(\sigma-1)^{-2}\eta^{-\frac{3}{2}}\cos\left(2\sigma^{\frac{1}{2}}\eta-\tfrac{3}{4}\pi\right), \quad (5.27)$$

as $\eta\to\infty$, where $\sigma>1$. A direct derivation of this result has also been carried out using the integral representation (5.20).

## 6. Asymptotic properties of the zeros of $L_n(u)$

The high-field polynomial $L_n(u)$ has a trivial zero at $u=0$, and $n-1$ simple non-trivial zeros $\{u_\nu(\sigma,n);\ \nu=1,2,\ldots,(n-1)\}$ which are all located in the real interval $u_c<u<1$. (A general proof of this last remarkable property has not yet been established.) We shall enumerate the non-trivial zeros in ascending order with

$$u_c < u_1(\sigma,n) < u_2(\sigma,n) < \ldots < u_{n-1}(\sigma,n) < 1. \quad (6.1)$$

To investigate the asymptotic properties of these zeros as $n\to\infty$ we first apply the transformation $\vartheta = \mathrm{i}\theta$ to the modified expansion (4.28). This procedure gives

$$L_n(u) \sim 2\pi^{\frac{1}{2}}\bar{a}(\sigma,\theta)\,(\bar{\zeta})^{\frac{1}{4}}\,n^{-\frac{7}{3}}\left[1+\sum_{m=1}^{\infty}\bar{E}_m(\sigma,\theta)\,n^{-2m}\right]\mathrm{Ai}\left(-\bar{\zeta}n^{\frac{2}{3}}+\bar{\epsilon}\right), \quad (6.2)$$

where
$$\bar{\epsilon} = \bar{\epsilon}(n,\sigma,\theta) = n^{-\frac{4}{3}}\sum_{m=0}^{\infty}\bar{H}_m(\sigma,\theta)\,n^{-2m}, \quad (6.3)$$

$$\bar{E}_m(\sigma,\theta) = E_m(\sigma,\mathrm{i}\theta), \quad (6.4)$$
$$\bar{H}_m(\sigma,\theta) = H_m(\sigma,\mathrm{i}\theta). \quad (6.5)$$

From equations (4.31)–(4.33) we readily find that

$$\bar{H}_0(\sigma,\theta) = \bar{B}_0(\sigma,\theta), \quad (6.6)$$
$$\bar{E}_1(\sigma,\theta) = \bar{A}_1(\sigma,\theta)+\tfrac{1}{2}[\bar{B}_0(\sigma,\theta)]^2\bar{\zeta}, \quad (6.7)$$
$$\bar{H}_1(\sigma,\theta) = \bar{B}_1(\sigma,\theta)-\bar{B}_0(\sigma,\theta)\,\bar{A}_1(\sigma,\theta)-\tfrac{1}{3}[\bar{B}_0(\sigma,\theta)]^3\bar{\zeta}, \quad (6.8)$$

where $\bar{\zeta}=\bar{\zeta}(\sigma,\theta)$ is defined in (5.3).

It is now evident from (6.2) that $L_n(u)$ will be asymptotically equal to zero when

$$\bar{\zeta}(\sigma,\theta_\nu) = |a_\nu|\,n^{-\frac{2}{3}}+\sum_{m=0}^{\infty}\bar{H}_m(\sigma,\theta_\nu)\,n^{-2m-2}, \quad (6.9)$$

Table 1. *Comparison of the exact values of the zeros* $\{u_\nu\ (11,\ 30);\ \nu = 1, 2, \ldots, 29\}$ *with the corresponding asymptotic values*

(The quantity $e_\nu(11, 30)$ is the difference between the exact value of $u_\nu(11, 30)$ and the asymptotic value of $u_\nu(11, 30)$ as determined from the formula (6.9).)

| $\nu$ | exact $u_\nu(11,\ 30)$ | $e_\nu(11,\ 30)$ | $\nu$ | exact $u_\nu(11,\ 30)$ | $e_\nu(11,\ 30)$ |
|---|---|---|---|---|---|
| 1 | 0.733 063 277 996 444 | $1.0\times10^{-10}$ | 16 | 0.948 220 396 494 102 | $6.0\times10^{-9}$ |
| 2 | 0.761 534 131 920 581 | $1.4\times10^{-10}$ | 17 | 0.955 410 053 876 478 | $8.2\times10^{-9}$ |
| 3 | 0.784 457 818 205 941 | $1.9\times10^{-10}$ | 18 | 0.962 025 254 025 866 | $1.2\times10^{-8}$ |
| 4 | 0.804 339 045 336 472 | $2.5\times10^{-10}$ | 19 | 0.968 080 115 354 309 | $1.7\times10^{-8}$ |
| 5 | 0.822 134 531 975 231 | $3.3\times10^{-10}$ | 20 | 0.973 586 541 034 947 | $2.4\times10^{-8}$ |
| 6 | 0.838 336 101 035 012 | $4.2\times10^{-10}$ | 21 | 0.978 554 580 816 165 | $3.7\times10^{-8}$ |
| 7 | 0.853 237 439 540 788 | $5.4\times10^{-10}$ | 22 | 0.982 992 712 932 553 | $5.8\times10^{-8}$ |
| 8 | 0.867 030 790 117 555 | $7.0\times10^{-10}$ | 23 | 0.986 908 063 673 323 | $9.7\times10^{-8}$ |
| 9 | 0.879 850 076 218 888 | $8.9\times10^{-10}$ | 24 | 0.990 306 575 299 621 | $1.7\times10^{-7}$ |
| 10 | 0.891 792 881 084 122 | $1.1\times10^{-9}$ | 25 | 0.993 193 125 039 559 | $3.3\times10^{-7}$ |
| 11 | 0.902 932 754 234 730 | $1.5\times10^{-9}$ | 26 | 0.995 571 582 367 334 | $7.2\times10^{-7}$ |
| 12 | 0.913 326 603 658 309 | $1.9\times10^{-9}$ | 27 | 0.997 444 744 885 551 | $1.9\times10^{-6}$ |
| 13 | 0.923 019 385 246 473 | $2.5\times10^{-9}$ | 28 | 0.998 813 898 638 104 | $6.3\times10^{-6}$ |
| 14 | 0.932 047 210 454 977 | $3.3\times10^{-9}$ | 29 | 0.999 676 570 095 532 | $3.5\times10^{-5}$ |
| 15 | 0.940 439 480 559 020 | $4.4\times10^{-9}$ | | | |

where $\nu = 1, 2, \ldots, n-1$ and $a_\nu$ denotes the $\nu$th negative zero of the Airy function $\mathrm{Ai}\,(z)$. (Note that all the zeros of $\mathrm{Ai}\,(z)$ lie on the negative real axis.) If the implicit transcendental equation (6.9) is solved for the quantity $\theta_\nu = \theta_\nu(\sigma, n)$, then the $\nu$th zero of $L_n(u)$ is given by

$$u_\nu(\sigma, n) \sim (\sigma-1)^2/[(\sigma^2+1)+2\sigma\cos\theta_\nu], \tag{6.10}$$

where $\nu = 1, 2, \ldots, n-1$. This procedure has been carried out for $n = 30$ and $q = 12$ by applying a direct iterative method to equation (6.9) with the coefficients $\bar{H}_0(\sigma, \theta_\nu)$ and $\bar{H}_1(\sigma, \theta_\nu)$ and a suitable initial value for $\theta_\nu(11, 30)$. The resulting asymptotic values for $u_\nu(11, 30)$, $(\nu = 1, 2, \ldots, 29)$ are compared with the corresponding exact values in table 1. We see that (6.9) gives very accurate approximations for the zeros $u_\nu(11, 30)$, $(\nu = 1, 2, \ldots, 29)$ especially for small values of $\nu$.

When $n \to \infty$ with $\nu$ fixed the quantity $u_\nu(\sigma, n)$ has a limiting value of $u_c$. Under these circumstances we can use (5.16) to obtain the leading-order asymptotic representation

$$[u_\nu(\sigma, n) - u_c]/u_c \sim 2^{\frac{4}{3}}\sigma^{\frac{1}{3}}(\sigma^2-1)^{-\frac{2}{3}}|a_\nu| n^{-\frac{2}{3}}, \tag{6.11}$$

as $n \to \infty$. Higher-order representations may be derived by first applying the Taylor series (4.22) and (4.34) with $\vartheta = \mathrm{i}\theta$ to equation (6.9). This procedure gives

$$X_\nu = Y_\nu \sum_{k=0}^{\infty} \zeta_k(\sigma)\,(-Y_\nu)^k - \sum_{m=0}^{\infty} (|a_\nu|^{-1}X_\nu)^{3m+3} \sum_{k=0}^{\infty} H_k^{(m)}(\sigma)\,(-Y_\nu)^k, \tag{6.12}$$

where
$$X_\nu = X_\nu(\sigma, n) = |a_\nu|\,[\tfrac{1}{4}\sigma(\sigma^2-1)\,n]^{-\frac{2}{3}}, \tag{6.13}$$

$$Y_\nu = Y_\nu(\sigma, \theta_\nu) = [\theta_\nu/(1+\sigma)]^2. \tag{6.14}$$

Next we use the implicit relation (6.12) to expand $Y_\nu$ in the form

$$Y_\nu = \sum_{k=0}^{\infty} \bar{C}_k(\sigma, \nu)\,(X_\nu)^{k+1}, \tag{6.15}$$

Table 2. *Values of the zeros $\eta_\nu(\sigma)$ for $\sigma = 11$*

| $\nu$ | $\eta_\nu(11)$ | $\nu$ | $\eta_\nu(11)$ |
|---|---|---|---|
| 1 | 0.539 466 247 662 236 | 6 | 2.947 296 356 688 207 |
| 2 | 1.032 882 640 695 021 | 7 | 3.422 625 022 029 556 |
| 3 | 1.515 577 449 518 671 | 8 | 3.897 551 031 310 762 |
| 4 | 1.994 386 226 228 526 | 9 | 4.372 200 486 420 704 |
| 5 | 2.471 349 751 590 382 | 10 | 4.846 652 017 405 834 |

where $\bar{C}_k(\sigma, \nu)$ is, in general, a polynomial in $\sigma$ and $|a_\nu|^{-3}$. There is sufficient data in Appendixes 1, 3 and 5 to determine the polynomials $\bar{C}_k(\sigma, \nu)$ for $k \leqslant 8$. We now use equations (3.12) and (6.14) to obtain the expansion

$$u_\nu(\sigma, n)/u_c \sim \left[1 + \frac{2\sigma}{(1+\sigma)^2} \sum_{m=1}^{\infty} (1+\sigma)^{2m} \frac{(-Y_\nu)^m}{(2m)!}\right]^{-1}. \tag{6.16}$$

Finally, the substitution of the series (6.15) in (6.16) yields the required formula

$$[u_\nu(\sigma, n) - u_c]/u_c \sim \sigma X_\nu \left[1 + \sum_{m=1}^{\infty} Q_m(\sigma, \nu)(X_\nu)^m\right], \tag{6.17}$$

as $n \to \infty$ with $\nu$ fixed, where the quantity $X_\nu = X_\nu(\sigma, n)$ is defined in equation (6.13). Expressions for the coefficients $Q_m(\sigma, \nu)$ in (6.17) are given in Appendix 6 for $m \leqslant 7$. The basic result (6.17) is consistent with the empirical numerical analysis of Majumdar (1974) and the predictions of scaling-law theory (Gaunt 1978).

We have evaluated the expansion (6.17) numerically for the particular case $\sigma = 11$ and $n = 30$ using the results in Appendix 6. In this manner we obtain the approximations

$$u_1(11, 30) \approx 0.733\,063\,278\,031,$$
$$u_2(11, 30) \approx 0.761\,534\,132\,484,$$
$$u_3(11, 30) \approx 0.784\,457\,819\,917.$$

These results for $\{u_\nu(11, 30); \nu = 1, 2, 3\}$ are in excellent agreement with the corresponding exact values given in table 1. For increasing values of $\nu = 4, 5, \ldots, 29$ there is a gradual decrease in the accuracy of the asymptotic approximation (6.17). In general one would expect (6.17) to give an accurate representation for $u_\nu(\sigma, n)$ provided that $n$ is sufficiently large and $1 \leqslant \nu \ll n$. It should be noted, however, that the expansion (6.17) breaks down completely in the limit $\sigma \to 1+$. When $\nu \gg 1$ we can use the standard asymptotic expansion for $a_\nu$ (Antosiewicz 1965) to simplify (6.17).

The dominant asymptotic behaviour of the zeros (6.1) which are close to the limit point $u = 1$ can be readily determined by considering the asymptotic relation (5.19) and equation (5.21). It is found that

$$u_{n-\nu}(\sigma, n) \sim 1 - [\eta_\nu(\sigma)/n]^2 + \ldots, \tag{6.18}$$

as $n \to \infty$, where $1 \leqslant \nu \ll n$ and $\eta_\nu(\sigma)$ is the $\nu$th positive zero of the scaling function $J(\sigma, \eta)$. The correction terms to the expansion (6.18) only involve even powers of $1/n$. Numerical values for the zeros $\{\eta_\nu(\sigma); \nu = 1, 2, \ldots\}$ have been calculated for the special case $\sigma = 11$ by applying the Newton–Raphson method to the series (5.24). We list the final results in table 2 for $\nu \leqslant 10$. It is now possible to use (6.18) and table 2 to derive the approximation

$$u_{29}(11, 30) \approx 0.999\,676\,644,$$

which is in good agreement with the corresponding exact value in table 1. We see from equation (5.23) that (6.18) also remains valid when $\sigma = 1$, with

$$\eta_\nu(1) = \tfrac{1}{2} j_{0,\nu}, \tag{6.19}$$

where $j_{0,\nu}$ denotes the $\nu$th positive zero of the Bessel function $J_0(z)$.

In the limit $n \to \infty$ the zeros $\{u_\nu(\sigma, n); \nu = 1, 2, \ldots, (n-1)\}$ form a dense quasi-continuum in the interval $(u_c, 1)$. The limiting density $\rho(\sigma, u)$ of these zeros can be determined by first using equations (3.12) and (5.3) to express the dominant part of the asymptotic formula (6.9) in the alternative form

$$nf(\sigma, u_\nu) \sim \tfrac{4}{3}|a_\nu|^{\frac{3}{2}}, \tag{6.20}$$

where
$$f(\sigma, u) = -(\sigma-1)\arccos\left[\frac{(\sigma-1)^2 - u(\sigma^2+1)}{2\sigma u}\right] + (\sigma+1)\arccos\left[\frac{(\sigma^2+1) - u(\sigma+1)^2}{2\sigma}\right]. \tag{6.21}$$

Next we substitute the asymptotic representation

$$|a_\nu|^{\frac{3}{2}} \sim \tfrac{3}{2}\pi\nu, \tag{6.22}$$

as $\nu \to \infty$, in (6.20) and differentiate the resulting expression with respect to $\nu$. In this manner we find that

$$\rho(\sigma, u) = n(2\pi)^{-1}(\partial f/\partial u). \tag{6.23}$$

Finally, the evaluation of the derivative in (6.23) yields the limiting density function

$$\rho(\sigma, u) = n(2\pi)^{-1}(\sigma+1)\,u^{-1}(u-u_c)^{\frac{1}{2}}(1-u)^{-\frac{1}{2}}, \tag{6.24}$$

as $n \to \infty$, where $u_c < u < 1$. For the special case $\sigma = 1$ the formula (6.24) gives the correct result for the one-dimensional Ising model Joyce (1990).

## 7. Mean-field limit

If we define $J \equiv J_0/q$, where $J_0$ is a positive constant, and take the limit $q \to \infty$ then the Bethe approximation reduces to the mean-field theory. (The introduction of the constant $J_0$ ensures that the ground-state energy of the system remains finite as $q \to \infty$.) In this mean-field limit we find from equation (2.19) and the expressions (2.28)–(2.31) that the first few coefficients $L_n(u)$ can be written in the form

$$L_1(K) = \exp(-2K), \tag{7.1}$$

$$L_2(K) = -\tfrac{1}{2}(1-4K)\exp(-4K), \tag{7.2}$$

$$L_3(K) = \tfrac{1}{3}(1-12K+24K^2)\exp(-6K), \tag{7.3}$$

$$L_4(K) = -\tfrac{1}{12}(3-72K+384K^2-512K^3)\exp(-8K), \tag{7.4}$$

where
$$K \equiv J_0/(k_B T). \tag{7.5}$$

For the general case it has been proved by one of us (Joyce, unpublished work) that

$$n^2 L_n(K) = (-1)^{n-1} L_{n-1}^{(1)}(4nK)\exp(-2nK), \tag{7.6}$$

where $L_n^{(1)}(x)$ denotes a generalized Laguerre polynomial (Szegö 1939).

The asymptotic behaviour of $L_n(K)$ for $1 < K < \infty$ can be determined by taking the limit $\sigma \to \infty$ in equation (4.10). To leading order this procedure gives

$$L_n(K) \sim 2^{-\frac{3}{2}} K^{-\frac{3}{4}}(K-1)^{-\frac{1}{4}} \zeta^{\frac{1}{4}} n^{-\frac{7}{3}} \operatorname{Ai}(\zeta n^{\frac{2}{3}}), \tag{7.7}$$




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as $n \to \infty$, where

$$\zeta = \zeta(K) = 3^{\frac{2}{3}}[K^{\frac{1}{2}}(K-1)^{\frac{1}{2}} - \operatorname{arccosh}(K^{\frac{1}{2}})]^{\frac{2}{3}}, \tag{7.8}$$

and $1 < K < \infty$. In a similar manner we find from equation (5.7) that

$$L_n(K) \sim 2^{-\frac{3}{2}} K^{-\frac{3}{4}} (1-K)^{-\frac{1}{4}} (\bar{\zeta})^{\frac{1}{4}} n^{-\frac{7}{3}} \operatorname{Ai}(-\bar{\zeta} n^{\frac{2}{3}}), \tag{7.9}$$

as $n \to \infty$, where

$$\bar{\zeta} = \bar{\zeta}(K) = 3^{\frac{2}{3}}[\arccos(K^{\frac{1}{2}}) - K^{\frac{1}{2}}(1-K)^{\frac{1}{2}}]^{\frac{2}{3}}, \tag{7.10}$$

and $0 < K < 1$. (It should be noted that in the mean-field limit the critical value of $K$ is $K_c = 1$.) These asymptotic representations for $L_n(K)$ can also be derived directly from (7.6) by using known results for the generalized Laguerre polynomial (Erdélyi 1960). In the critical region $|K-1| \ll 1$ the uniform asymptotic formulae (7.7) and (7.9) reduce to

$$L_n(K) \sim 2^{-\frac{4}{3}} n^{-\frac{7}{3}} \operatorname{Ai}[2^{\frac{2}{3}}(K-1)\, n^{\frac{2}{3}}], \tag{7.11}$$

as $n \to \infty$, and $K \to 1\pm$. It is, of course, possible to obtain higher-order asymptotic representations for $L_n(K)$. For example, by taking the limit $\sigma \to \infty$ in (4.25) we find that

$$L_n(K_c) \sim 2^{-\frac{4}{3}} n^{-\frac{7}{3}} [\operatorname{Ai}(0)\{1 + \tfrac{2}{225} n^{-2} + \ldots\} + \tfrac{3}{70} 2^{\frac{2}{3}} \operatorname{Ai}'(0)\, n^{-\frac{4}{3}} \{1 - \tfrac{724}{14625} n^{-2} + \ldots\}], \tag{7.12}$$

as $n \to \infty$.

It follows from the theory of orthogonal polynomials (Szegö 1939) that the generalized Laguerre polynomial $L_n^{(1)}(x)$ has $n$ simple zeros $\{x_\nu(n); \nu = 1, 2, \ldots, n\}$ which are all located in the real interval $0 < x < 4(n+1)$. We shall enumerate these zeros in ascending order with

$$x_1(n) < x_2(n) < \ldots < x_n(n). \tag{7.13}$$

From these results and equation (7.6) it is clear that $L_n(K)$ has $n-1$ simple zeros

$$K_\nu(n) = (4n)^{-1} x_\nu(n-1), \tag{7.14}$$

where $\nu = 1, 2, \ldots, (n-1)$, which all lie in the real interval $0 < K < 1$. Accurate values for the zeros $x_\nu(n)$ have been given by Rabinowitz & Weiss (1959) for $n = 4, 8, 12$ and 16.

The asymptotic behaviour of the zeros $K_\nu(n)$ which are close to $K_c$ can be established by taking the mean-field limit in equation (6.17). This procedure yields the expansion

$$1 - K_{n-\nu}(n) \sim |a_\nu|\,(2n)^{-\frac{2}{3}} \left[1 - \sum_{m=1}^{\infty} \Lambda_m(\nu)\, |a_\nu|^m (2n)^{-2m/3}\right], \tag{7.15}$$

as $n \to \infty$, with $1 \leqslant \nu \ll n$. Expressions for the coefficients $\Lambda_m(\nu)$ in (7.15) are given in Appendix 7 for $m \leqslant 7$. In order to determine the asymptotic behaviour of the zeros $K_\nu(n)$ which are close to $K = 0$ we take the mean-field limit in (5.19) and apply the standard integral representation

$$J_1(z) = \frac{2z}{\pi} \int_0^{\pi/2} \cos(z\cos\theta) \sin^2\theta \, d\theta, \tag{7.16}$$

where $J_1(z)$ denotes a Bessel function of the first kind. Hence we obtain

$$L_n(K) \sim \frac{(-1)^{n-1}}{2K^{\frac{1}{2}} n^2} J_1(4K^{\frac{1}{2}} n), \tag{7.17}$$





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as $n \to \infty$ and $K \to 0+$. It follows directly from (7.17) that

$$K_\nu(n) \sim j_{1,\nu}^2 (4n)^{-2} + \ldots, \tag{7.18}$$

as $n \to \infty$, with $1 \leqslant \nu \ll n$, where $j_{1,\nu}$ denotes the $\nu$th positive zero of the Bessel function $J_1(z)$. The extensive work of Tricomi (1949) on the asymptotic properties of generalized Laguerre polynomials leads to the higher-order representation

$$K_\nu(n) \sim j_{1,\nu}^2 (4n)^{-2} [1 + \tfrac{1}{3} j_{1,\nu}^2 (4n)^{-2} + \ldots], \tag{7.19}$$

as $n \to \infty$, with $1 \leqslant \nu \ll n$.

Finally, the application of the mean-field limit to equation (6.24) yields the simple formula

$$\rho(K) = n(2/\pi) K^{-\frac{1}{2}} (1-K)^{\frac{1}{2}}, \tag{7.20}$$

where $\rho(K)$ is the density of the zeros $\{K_\nu(n); \nu = 1, 2, \ldots, (n-1)\}$ in the interval $(0, 1)$ as $n \to \infty$.

It is hoped to give a more detailed direct analysis of the mean-field case and a proof of the formula (7.6) in a further publication.

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## Appendix 1. Coefficients $\zeta_k(\sigma)$ in expansion (4.22)

$$\zeta_0(\sigma) = 1,$$

$$\zeta_1(\sigma) = \tfrac{1}{30}\sigma(s_1 - 10),$$

$$\zeta_2(\sigma) = \tfrac{1}{25\,200}\sigma^2(13s_2 - 980s_1 + 4206),$$

$$\zeta_3(\sigma) = \tfrac{1}{252\,000}\sigma^3(s_3 - 510s_2 + 9023s_1 - 25\,220),$$

$$\zeta_4(\sigma) = \tfrac{1}{6\,985\,440\,000}\sigma^4(139s_4 - 440\,440s_3 + 24\,198\,676s_2 - 216\,120\,520s_1 + 469\,124\,674),$$

$$\zeta_5(\sigma) = \tfrac{1}{2\,724\,321\,600\,000}\sigma^5(203s_5 - 3\,621\,670s_4 + 554\,820\,495s_3 - 12\,216\,489\,480s_2 + 71\,705\,103\,270s_1 - 132\,533\,683\,620),$$

$$\zeta_6(\sigma) = \tfrac{1}{95\,351\,256\,000\,000}\sigma^6(s_6 - 1\,935\,360s_5 + 788\,781\,206s_4 - 38\,184\,064\,800s_3 + 485\,334\,126\,815s_2 - 2\,137\,854\,437\,600s_1 + 3\,546\,706\,131\,220),$$

$$\zeta_7(\sigma) = \tfrac{1}{48\,629\,140\,560\,000\,000}\sigma^7(107s_7 - 11\,424\,170s_6 + 12\,127\,675\,769s_5 - 1\,213\,520\,646\,820s_4 + 30\,019\,362\,356\,947s_3 - 261\,751\,756\,885\,750s_2 + 935\,747\,888\,636\,745s_1 - 1\,438\,480\,569\,166\,200).$$

## Appendix 2. Coefficients $A_k^{(1)}(\sigma)$ in expansion (4.23)

$$A_0^{(1)}(\sigma) = \tfrac{1}{3600}\sigma^3(2s_3 - 270s_2 - 1669s_1 - 2790),$$

$$A_1^{(1)}(\sigma) = -\tfrac{1}{616\,000}\sigma^4(57s_4 - 13\,860s_3 - 172\,247s_2 - 561\,330s_1 - 805\,858),$$

$$A_2^{(1)}(\sigma) = \tfrac{1}{720\,720\,000}\sigma^5(11\,868s_5 - 2\,665\,455s_4 - 38\,977\,160s_3 - 180\,029\,070s_2 - 411\,044\,570s_1 - 534\,614\,730),$$

$$A_3^{(1)}(\sigma) = -\tfrac{1}{1\,816\,214\,400\,000}\sigma^6(4\,422\,469s_6 - 938\,504\,700s_5 - 15\,789\,649\,891s_4 - 92\,960\,471\,100s_3 - 289\,702\,264\,035s_2 - 549\,169\,924\,800s_1 - 675\,238\,158\,350).$$

## Appendix 3. Coefficients $B_k^{(0)}(\sigma)$ in expansion (4.24)

$$B_0^{(0)}(\sigma) = \tfrac{1}{280}\sigma^2(3s_2+105s_1+206),$$

$$B_1^{(0)}(\sigma) = -\tfrac{1}{3600}\sigma^3(4s_3+135s_2+262s_1+270),$$

$$B_2^{(0)}(\sigma) = \tfrac{1}{232\,848\,000}\sigma^4(26\,287s_4+949\,025s_3+4\,551\,698s_2+5\,118\,575s_1+3\,245\,822),$$

$$B_3^{(0)}(\sigma) = -\tfrac{1}{18\,162\,144\,000}\sigma^5(209\,746s_5+7\,814\,365s_4+49\,461\,364s_3 +165\,240\,920s_2+123\,390\,362s_1+12\,993\,110),$$

$$B_4^{(0)}(\sigma) = \tfrac{1}{726\,485\,760\,000}\sigma^6(855\,059s_6+33\,450\,837s_5+261\,528\,066s_4 +963\,600\,785s_3+3\,122\,879\,533s_2+1\,560\,582\,170s_1-1\,295\,586\,948),$$

$$B_5^{(0)}(\sigma) = -\tfrac{1}{38\,903\,312\,448\,000\,000}\sigma^7(4\,651\,223\,096s_7+190\,894\,563\,235s_6 +1\,798\,775\,523\,922s_5+8\,392\,429\,860\,410s_4+20\,267\,774\,931\,066s_3 +84\,069\,131\,634\,525s_2+28\,290\,848\,235\,260s_1-70\,332\,679\,325\,300).$$

## Appendix 4. Coefficients $B_k^{(1)}(\sigma)$ in expansion (4.24)

$$B_0^{(1)}(\sigma) = -\tfrac{1}{32\,760\,000}\sigma^5(1086s_5+33\,150s_4+4\,011\,430s_3+30\,942\,600s_2 +88\,221\,985s_1+122\,512\,650),$$

$$B_1^{(1)}(\sigma) = \tfrac{1}{1\,552\,320\,000}\sigma^6(9987s_6+589\,785s_5+69\,246\,122s_4+661\,434\,550s_3 +2\,543\,061\,355s_2+5\,313\,468\,475s_1+6\,726\,215\,440),$$

$$B_2^{(1)}(\sigma) = -\tfrac{1}{10\,291\,881\,600\,000}\sigma^7(12\,005\,529s_7+898\,177\,320s_6+102\,669\,170\,228s_5 +1\,159\,827\,471\,545s_4+5\,525\,008\,930\,484s_3+14\,851\,116\,077\,800s_2 +25\,703\,282\,239\,015s_1+30\,637\,747\,717\,150),$$

$$B_3^{(1)}(\sigma) = \tfrac{1}{82\,129\,215\,168\,000\,000}\sigma^8(15\,741\,056\,484s_8+1\,309\,038\,392\,025s_7 +147\,819\,155\,295\,892s_6+1\,926\,816\,851\,124\,900s_5 +11\,141\,680\,413\,455\,547s_4+37\,190\,071\,296\,634\,900s_3 +80\,753\,202\,156\,926\,654s_2+123\,663\,928\,844\,824\,375s_1 +141\,476\,414\,843\,946\,030).$$

## Appendix 5. Coefficients $H_k^{(1)}(\sigma)$ in expansion (4.34)

$$H_0^{(1)}(\sigma) = -\tfrac{1}{9\,360\,000}\sigma^5(366s_5+3900s_4+840\,205s_3+6\,621\,225s_2+18\,983\,785s_1 +26\,397\,150),$$

$$H_1^{(1)}(\sigma) = \tfrac{1}{97\,796\,160\,000}\sigma^6(826\,601s_6+15\,071\,805s_5+3\,085\,150\,156s_4+28\,832\,630\,400s_3 +108\,720\,988\,965s_2+224\,411\,904\,675s_1+282\,897\,175\,820),$$

$$H_2^{(1)}(\sigma) = -\tfrac{1}{1\,296\,777\,081\,600\,000}\sigma^7(2\,103\,829\,834s_7+49\,307\,414\,720s_6+9\,040\,407\,603\,063s_5 +100\,223\,811\,466\,945s_4+469\,761\,360\,635\,639s_3+1\,247\,843\,285\,160\,300s_2 +2\,142\,736\,841\,210\,440s_1+2\,546\,786\,643\,866\,150).$$

## Appendix 6. Coefficients $Q_m(\sigma,\nu)$ in expansion (6.17)

$$Q_1(\sigma,\nu) = -\tfrac{1}{20}\sigma(s_1-10),$$

$$Q_2(\sigma,\nu) = -\tfrac{1}{2800}\sigma^2[(3s_2+140s_1-494)-(30s_2+1050s_1+2060)\,|a_\nu|^{-3}],$$

$$\begin{aligned}Q_3(\sigma,\nu) = -\tfrac{1}{504\,000}\sigma^3[&(23s_3-90s_2+12\,569s_1-27\,180)\\ &-(20s_3+5400s_2+188\,060s_1+370\,800)\,|a_\nu|^{-3}],\end{aligned}$$

$$\begin{aligned}Q_4(\sigma,\nu) = -\tfrac{1}{388\,080\,000}\sigma^4[&(947s_4-3080s_3-279\,212s_2+3\,378\,760s_1-6\,050\,318)\\ &+(-1700s_4+192\,500s_3+5\,169\,950s_2-57\,557\,500s_1-128\,481\,700)\,|a_\nu|^{-3}],\end{aligned}$$

$$\begin{aligned}Q_5(\sigma,\nu) = -\tfrac{1}{67\,267\,200\,000}\sigma^5[&(9879s_5-27\,950s_4-695\,605s_3-29\,621\,800s_2\\ &+171\,013\,790s_1-281\,487\,700)+(-30\,300s_5+611\,000s_4+61\,727\,500s_3\\ &+1\,176\,214\,000s_2-1\,847\,966\,000s_1-6\,267\,794\,000)\,|a_\nu|^{-3}\\ &+(2\,215\,620s_5-3\,861\,000s_4+5\,241\,000\,600s_3+40\,460\,706\,000s_2\\ &+114\,401\,723\,700s_1+158\,363\,634\,000)\,|a_\nu|^{-6}],\end{aligned}$$

$$\begin{aligned}Q_6(\sigma,\nu) = -\tfrac{1}{63\,567\,504\,000\,000}\sigma^6[&(604\,523s_6-1\,528\,380s_5-21\,000\,362s_4+119\,618\,100s_3\\ &-10\,456\,192\,155s_2+42\,740\,296\,200s_1-65\,165\,374\,540)+(-2\,837\,800s_6\\ &+19\,073\,250s_5+961\,283\,575s_4+14\,529\,296\,250s_3+542\,670\,649\,500s_2\\ &-81\,089\,977\,500s_1-1\,366\,830\,193\,750)\,|a_\nu|^{-3}+(180\,824\,435s_6\\ &+2\,066\,733\,900s_5+1\,297\,968\,109\,735s_4+18\,733\,396\,200\,750s_3\\ &+94\,895\,108\,894\,025s_2+231\,484\,291\,557\,750s_1\\ &+308\,031\,394\,967\,450)\,|a_\nu|^{-6}],\end{aligned}$$

$$\begin{aligned}Q_7(\sigma,\nu) = -\tfrac{1}{17\,290\,361\,088\,000\,000}\sigma^7[&(11\,192\,989s_7-25\,744\,630s_6-250\,790\,577s_5\\ &+1\,202\,217\,220s_4+58\,375\,335\,269s_3-841\,918\,809\,450s_2\\ &+2\,808\,180\,349\,615s_1-4\,032\,517\,942\,600)\\ &+(-74\,804\,840s_7+182\,056\,400s_6+14\,440\,952\,520s_5\\ &-147\,488\,069\,600s_4-3\,106\,416\,369\,640s_3+43\,405\,320\,414\,000s_2\\ &+9\,244\,660\,894\,600s_1-89\,230\,802\,920\,000)\,|a_\nu|^{-3}\\ &+(9\,901\,958\,656s_7+19\,883\,972\,480s_6+71\,166\,592\,304\,592s_5\\ &+1\,035\,789\,381\,594\,880s_4+7\,052\,340\,272\,301\,776s_3\\ &+26\,529\,214\,867\,027\,200s_2+57\,391\,969\,879\,492\,960s_1\\ &+73\,895\,444\,874\,809\,600)\,|a_\nu|^{-6}].\end{aligned}$$

## Appendix 7. Coefficients $\Lambda_m(\nu)$ in expansion (7.15)

$$\Lambda_1(\nu) = \tfrac{1}{5},$$

$$\Lambda_2(\nu) = \tfrac{3}{175}(1-10|a_\nu|^{-3}),$$

$$\Lambda_3(\nu) = \tfrac{1}{7875}(23-20|a_\nu|^{-3}),$$

$$\Lambda_4(\nu) = \tfrac{2}{3\,031\,875}(947-1700|a_\nu|^{-3}),$$

$$\Lambda_5(\nu) = \tfrac{1}{21\,896\,875}(3293-10\,100|a_\nu|^{-3}+738\,540|a_\nu|^{-6}),$$




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$$A_6(\nu) = \frac{4}{62\,077\,640\,625}(604\,523 - 2\,837\,800|a_\nu|^{-3} + 180\,824\,435|a_\nu|^{-6}),$$

$$A_7(\nu) = \frac{1}{1\,055\,319\,890\,625}(11\,192\,989 - 74\,804\,840|a_\nu|^{-3} + 9\,901\,958\,656|a_\nu|^{-6}).$$

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